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Fourier Analytic Approach to Quantum Estimation of Group Action

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Abstract: This article proposes a unified method to estimation of group action by using the inverse Fourier transform of the input state. The method can be applied to the non-commutative group as well as the commutative group. We show the square speed up phenomena of the estimation of the shift parameter in the energy constraint under the bosonic system.

1. Introduction

In quantum theory, it is often that the dynamics can be described by a projective unitary representation of a group. In this case, the unitary acting on the real quantum system reflects important physical parameters. Therefore, we can estimate these physical parameters by estimating the true unitary among a given projective unitary representation of a group. Indeed, it is known that estimation of unitary has a square speed up over the state estimation in quantum case. However, only the case of projective unitary representation has been solved[4, 5,9,6,11]. Other case of estimation of unitaries has not been solved while their Fisher information has been calculated[16]. Indeed, several researchers consider that the Fisher information describes the attainable limit of the precision of the estimation of unitary[17,18,19,20,21,22]. However, as was pointed in [14,13], it does not give the attainable bound of precision of the estimation of unitary.

The first studies [4,5] treated the phase estimation, which is essentially the estimation of the representation of $\mathrm{U}(1)$. Next, the estimation of $\mathrm{SU}(2)$ was studied [9,6,11]. Chiribella et al [7] established a general theory of estimation of unitary representation of a compact group. On the other hand, Imai et al [12] treated phase estimation by using Fourier analysis. This method can be extended to the case with energy constraint[13].

In this paper, we focus on the Fourier analytic approach as a unified approach to treat the non-compact case as well as the compact case. Employing the approach, we derive a general minimum error formula for the estimation of group

action. In this formula, the minimum error can be written as the minimum of the average error under the distribution by the inverse Fourier transform of the input pure state. This formula holds even in the projective unitary representation case.

Using the obtained general formula, we derive the optimal discrimination formula with the finite group case. The formula with the representation case has been shown by [7,10], and that with the projective representation case by [15]. Based the general formula, we derive a general formula for estimation of compact group[7,8]. Further, we apply our formula to the non-compact case. In this case, it is natural to impose the energy constraint for the input state. We also show that for this optimization, we can restrict the input state to pure states under some additional condition. Then, we apply the argument to the case of $\mathbb{R}$, $\mathbb{Z}$ and the Heisenberg representation. Since Heisenberg representation describes the bosonic system, the example is physically important.

The remaining parts are organized as follows. In Section 2, we give a formulation of the estimation of unknown group action. In this section, we also derive a general formula for minimum error as Theorem 3. In Section 3, we describe the minimization problem in the term of the Fourier analysis. In Section 4, we apply our general formula to the finite group case. In Section 5, we apply our general formula to the compact group case. In Section 6, we apply our general formula to the non-compact group case. As typical examples, we treat $\mathbb{R}$, $\mathbb{Z}$ and the Heisenberg representation.

2. General setting

We focus on a group G acting on the Hilbert space $\mathcal{H}$ of our interest. That is, we treat a projective unitary representation f of G over $\mathcal{H}$. Our aim is estimating the unknown unitary $f(g)$ under the assumption that $g \in G$. For this purpose, we can choose the input state ρ and the output measurement, which is described by the POVM M over the Hilbert space $\mathcal{H}$. Since the aim of the measurement is the estimation of the element of $g \in G$, the POVM M takes values in the group G. We describe the set of the above kinds of POVMs by $\mathcal{M}(G)$. Hence, our estimator is given as a pair of an input state ρ and a POVM M. We also allow to input a state entangled with another system as Fig. 1. This formulation covers the case as Fig. 2 when we choose the input state ρ_i with the probability p_i for $i = 1, \ldots, l$ and choose the output POVM M_i depending on the input state ρ_i. When the original input system is $\mathcal{H}'$, the above case can be described by the case when the input is $\sum_i p_i \rho_i \otimes |i\rangle\langle i|$ on $\mathcal{H}' \otimes \mathbb{C}^l$ and the POVM $M(\hat{g}) := \sum_i M_i(\hat{g}) \otimes |i\rangle\langle i|$ on $\mathcal{H}' \otimes \mathbb{C}^l$. Hence, the setting in Fig. 2 is included in the setting in Fig. 1.

In order to treat this problem, we focus on the risk function R depending on the true g and the estimate $\hat{g}$. Then, when the true is g, the average error is given as

$$\mathcal{D}_{R,g}(\rho, M) := \int_G R(g, \hat{g}) \mathrm{Tr}\, f(g) \rho f(g)^\dagger M(d\hat{g}). \tag{1}$$

Given a prior distribution ν for g over G, we can define the Bayesian error:

$$\mathcal{D}_{R,\nu}(M) := \int_G \mathcal{D}_{R,g}(M) \nu(dg). \tag{2}$$

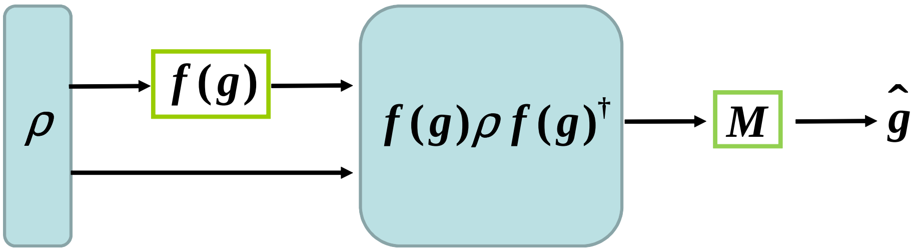

Fig. 1. Strategy for estimating the unknown group action g with an entangled input

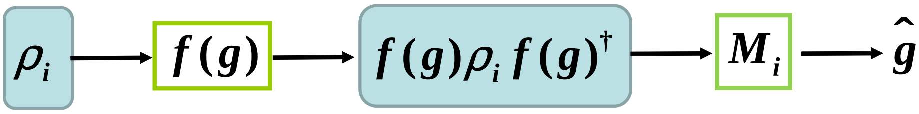

Fig. 2. Stochastic strategy for estimating the unknown group action g

Hence, our aim is finding a pair of the input state ρ and POVM $M \in \mathcal{M}(G)$ minimizing $\mathcal{D}_{R,\nu}(\rho, M)$.

As an alternative criterion, we optimize the worst case as

$$\mathcal{D}_R(M) := \max_G \mathcal{D}_{R,g}(M), \tag{3}$$

which is called the mini-max criterion.

Since the difference between g and $\hat{g}$ is thought to be the same as that between $g'g$ and $g'\hat{g}$, we assume the left invariant condition in the following:

$$R(g, \hat{g}) = R(g'g, g'\hat{g}), \quad \forall g, \hat{g}, g' \in G. \tag{4}$$

According to Holevo[1], as an important class of POVMs, we introduce a covariant POVM. In the original formulation, he treats the estimation of a homogeneous space. Since the group with the left action can be regarded as a homogeneous space, we can apply his general method to our problem. Hence, the right invariance in (4) is not needed for its application. A POVM M taking values in G is called covariant concerning the projective representation f when

$$f(g)M(B)f(g)^\dagger = M(gB). \tag{5}$$

Holevo[1] defined the concept for a general homogeneous space. The group G can be regarded as a special case of homogeneous spaces. We describe the set of covariant POVMs by $\mathcal{M}_{\mathrm{cov}}(G)$. For any covariant POVM $M \in \mathcal{M}_{\mathrm{cov}}(G)$, the average error $\mathcal{D}_{R,g}(\rho, M)$ does not depend on the true g. Hence, we obtain

$$\mathcal{D}_{R,g}(\rho, M) = \mathcal{D}_{R,\nu}(\rho, M) = \mathcal{D}_R(\rho, M). \tag{6}$$

When G is compact, there exists an invariant probability measure μ_G. Then, we obtain the following theorem, which is called quantum Hunt-Stein theorem[1].

Theorem 1 *When the risk function R is invariant and G is compact, we obtain*

$$\min_{M\in\mathcal{M}(G)} \mathcal{D}_{R,\mu_G}(\rho,M) = \min_{M\in\mathcal{M}(G)} \mathcal{D}_R(\rho,M) \tag{7}$$

$$= \min_{M\in\mathcal{M}_{\mathrm{cov}}(G)} \mathcal{D}_{R,\mu_G}(\rho,M) = \min_{M\in\mathcal{M}_{\mathrm{cov}}(G)} \mathcal{D}_R(\rho,M). \tag{8}$$

However, when G is not compact, it has no invariant probability measure. In this case, the above theorem can be generalized to the following way[2,3].

Theorem 2 *When the risk function R is left invariant and G is locally compact, we obtain*

$$\min_{M\in\mathcal{M}(G)} \mathcal{D}_{R,\mu_G}(\rho,M) = \min_{M\in\mathcal{M}_{\mathrm{cov}}(G)} \mathcal{D}_R(\rho,M). \tag{9}$$

Hence, in the following, in order to treat our problem without the compactness condition, we treat the minimization

$$\min_{\rho\in\mathcal{S}(\mathcal{H})} \min_{M\in\mathcal{M}_{\mathrm{cov}}(G)} \mathcal{D}_R(\rho,M), \tag{10}$$

where $\mathcal{S}(\mathcal{H})$ is the set of densities on $\mathcal{H}$. That is, we can restrict our measurement into covariant measurements without loss of generality. Given an input mixed state $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$, any measurement M satisfies

$$\mathcal{D}_{R,g}(\rho,M) = \sum_i p_i \mathcal{D}_{R,g}(|\phi_i\rangle\langle\phi_i|,M). \tag{11}$$

Hence, any covariant measurement M satisfies

$$\mathcal{D}_R(\rho,M) = \sum_i p_i \mathcal{D}_R(|\phi_i\rangle\langle\phi_i|,M). \tag{12}$$

Thus, we can restrict our input state into pure states without loss of generality.

Next, we characterize covariant POVMs. For this purpose, we assume that G has the left invariant measure μ_G. because we regard the group G as the homogeneous space concerning the left action. When G is compact, μ_G is chosen to be the probability measure. It is known that any covariant measurement M can be described by using a positive semi-definite operator T such that [1]

$$M(B) = \int_B f(g)Tf(g)^\dagger \mu_G(dg). \tag{13}$$

Conversely, the above kind of operator T satisfies

$$I = \int_G f(g)Tf(g)^\dagger \mu_G(dg). \tag{14}$$

When a positive semi-definite T satisfies (14), it gives a covariant measurement by (13), which is denoted by M_T.

In order to characterize a positive semi-definite T satisfying (14), we make the irreducible decomposition of $\mathcal{H}$ concerning the projective representation f:

$$\mathcal{H} = \oplus_{\lambda\in S}\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda \tag{15}$$

where $\hat{G}[\mathcal{L}]$ is the set of symbols of irreducible projective representation of G with the factor system $\mathcal{L}$, S is a subset of $\hat{G}[\mathcal{L}]$, $\mathcal{U}_\lambda$ is the irreducible space corresponding to $\lambda \in \hat{G}[\mathcal{L}]$, and $\mathcal{V}_\lambda$ is the space describing the multiplicity of the irreducible space $\mathcal{U}_\lambda$. That is, the group G acts only on $\mathcal{U}_\lambda$ but not on $\mathcal{V}_\lambda$.

When G is compact, all of irreducible space $\mathcal{U}_\lambda$ is finite-dimensional. When G is not compact, there are finite-dimensional irreducible spaces $\mathcal{U}_\lambda$. In this case, we define the generalized dimension $\dim_G \mathcal{U}_\lambda$ by

$$I = \dim_G \mathcal{U}_\lambda \int_G f(g)\rho f(g)^\dagger \mu_G(dg), \tag{16}$$

where ρ is an arbitrary state. In the following, we assume that the generalized dimension $\dim_G \mathcal{U}_\lambda$ is finite for any $\lambda \in S$.

For a pure state $|\phi\rangle\langle\phi|$, the family of output states $\{f(g)|\phi\rangle\langle\phi|f(g)^\dagger\}_{g\in G}$ belongs to a subspace $\oplus_{\lambda\in S}\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda'$, where the dimension $\mathcal{V}_\lambda'$ is $\min\{\dim \mathcal{V}_\lambda, \dim \mathcal{U}_\lambda\}$. Hence, in the following, we assume that $\dim \mathcal{V}_\lambda \le \dim \mathcal{U}_\lambda$. We a typical case, we often focus on the following representation space for a subset $S \subset \hat{G}[\mathcal{L}]$:

$$\mathcal{H}_S := \oplus_{\lambda\in S}\mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*. \tag{17}$$

In order to give a typical covariant POVM, we define a vector $|\mathcal{I}_\lambda\rangle := \sum_i |e_i\rangle\langle e_i| \in \mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*$ for CONS $\{e_i\}$ of $\mathcal{U}_\lambda$. Then, we define $|\mathcal{I}\rangle := \sum_\lambda \sqrt{d_\lambda}|\mathcal{I}_\lambda\rangle$. Here, the group G acts only on the first space $\mathcal{U}_\lambda$ under the representation space $\mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*$.

Then, we obtain the following theorem.

Theorem 3 *Assume all of the above conditions. Given an inclusion $\mathcal{V}_\lambda \subset \mathcal{U}_\lambda^*$, for any input pure state $|\phi\rangle\langle\phi|$ on $\mathcal{H}$ and any positive semi-definite T satisfying (14), there exists a mixed input state ρ on $\mathcal{H}$ such that*

$$\mathcal{D}_R(|\phi\rangle\langle\phi|, M_T) = \mathcal{D}_R(\rho, M_{|\mathcal{I}\rangle\langle\mathcal{I}|}). \tag{18}$$

Proof: First, we make a decomposition of the operator T as $T = \sum_k |\eta_k\rangle\langle\eta_k|$. In the following, we use the notations $|\phi\rangle = \oplus_\lambda|\phi_\lambda\rangle$, $|\eta_k\rangle = \oplus_\lambda|\eta_{k,\lambda}\rangle$, $|x_k\rangle := \oplus_\lambda \frac{1}{\sqrt{d_\lambda}}|\phi_\lambda\eta_{k,\lambda}^\dagger\rangle$. The output $\hat{g}$ satisfies the following distribution.

$$\begin{aligned}
&\sum_k |\langle\eta_k|f(\hat{g}^{-1})|x\rangle|^2\mu_G(d\hat{g}) = \sum_k |\sum_{\lambda\in S} \operatorname{Tr} \eta_{k,\lambda}^\dagger f_\lambda(\hat{g}^{-1})\phi_\lambda|^2 \mu_G(d\hat{g}) \\
=&\sum_k |\sum_{\lambda\in S} \operatorname{Tr} f_\lambda(\hat{g}^{-1})\phi_\lambda\eta_{k,\lambda}^\dagger|^2\mu_G(d\hat{g}) = \sum_k |\langle\mathcal{I}|f(\hat{g}^{-1})|x_k\rangle|^2\mu_G(d\hat{g}).
\end{aligned}$$

Let P_λ be the projection to $\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$ We choose an arbitrary operator $A \otimes B$ on $\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$. Due to (14), we obtain

$$\begin{aligned}
&\operatorname{Tr} A \operatorname{Tr} B = \operatorname{Tr} A \otimes B P_\lambda = \operatorname{Tr} A \otimes B P_\lambda \int_G f(g) T f(g)^\dagger \mu_G(dg) P_\lambda \\
=&\operatorname{Tr} (\int_G f(g)_\lambda^\dagger A f_\lambda(g)\mu_G(dg)) \otimes B(P_\lambda T P_\lambda) = \operatorname{Tr} \frac{\operatorname{Tr} A}{d_\lambda} I_{\mathcal{U}_\lambda} \otimes B(P_\lambda T P_\lambda) \\
=&\frac{\operatorname{Tr} A}{d_\lambda}\operatorname{Tr} B \operatorname{Tr}_{\mathcal{U}_\lambda}(P_\lambda T P_\lambda) = \frac{\operatorname{Tr} A}{d_\lambda}\operatorname{Tr} B \operatorname{Tr}_{\mathcal{U}_\lambda}(P_\lambda T P_\lambda)
\end{aligned}$$

That is,

$$\frac{1}{d_\lambda} \mathrm{Tr}_{\mathcal{U}_\lambda} (P_\lambda T P_\lambda) = I_{\mathcal{V}_\lambda}.$$

Therefore,

$$\sum_k \mathrm{Tr}\, |x_k\rangle\langle x_k| = \sum_k \sum_\lambda \frac{1}{d_\lambda} \mathrm{Tr}\, \eta_{k,\lambda} \phi_\lambda^\dagger \phi_\lambda \eta_{k,\lambda}^\dagger = \sum_\lambda \mathrm{Tr}\, \phi_\lambda^\dagger \phi_\lambda \sum_k \frac{1}{d_\lambda} \eta_{k,\lambda}^\dagger \eta_{k,\lambda}$$
$$= \sum_\lambda \mathrm{Tr}\, \phi_\lambda^\dagger \phi_\lambda I_{\mathcal{V}_\lambda} = \mathrm{Tr}\, |x\rangle\langle x| = 1.$$

Hence, we obtain

$$\mathcal{D}_R(|\phi\rangle\langle\phi|, M_T) = \mathcal{D}_R(\sum_k |x_k\rangle\langle x_k|, M_{|\mathcal{I}\rangle\langle\mathcal{I}|}).$$

■

Since any mixed input state ρ on $\mathcal{H}$ can be written as a convex combination of pure states on $\mathcal{H}$,

$$\min_{\rho \in \mathcal{S}(\mathcal{H})} \mathcal{D}_R(\rho, M_{|\mathcal{I}\rangle\langle\mathcal{I}|}) = \min_{|\phi\rangle \in \mathcal{H}} \mathcal{D}_R(|\phi\rangle\langle\phi|, M_{|\mathcal{I}\rangle\langle\mathcal{I}|}). \tag{19}$$

In the following, for simplicity, we denote $\mathcal{D}_R(|\phi\rangle\langle\phi|, M_{|\mathcal{I}\rangle\langle\mathcal{I}|})$ by $\mathcal{D}_R(|\phi\rangle)$.

That is, we obtain

$$\min_{\rho \in \mathcal{S}(\mathcal{H})} \min_{M \in \mathcal{M}_{\mathrm{cov}}(G)} \mathcal{D}_R(\rho, M) = \min_{|\phi\rangle \in \mathcal{H}} \min_T \mathcal{D}_R(|\phi\rangle\langle\phi|, M_T) = \min_{|\phi\rangle \in \mathcal{H}} \mathcal{D}_R(|\phi\rangle).$$

3. Relation to Fourier transform

In this section, in order to treat the relation to Fourier transform, we prepare notations. In the Fourier analysis for a general group, depending on the factor system $\mathcal{L}$, we define the space $L^2(\hat{G}[\mathcal{L}]) := \sum_{\lambda \in \hat{G}[\mathcal{L}]} \mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*$. Here, we can identify the space $\mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*$ with the space of the Hilbert Schmidt operators on $\mathcal{U}_\lambda$.

The Fourier transform with the factor system $\mathcal{L}$ is defined as a unitary map $\mathcal{F}_\mathcal{L}$ from $L^2(G)$ to $L^2(\hat{G}[\mathcal{L}])$ as follows. Given $\phi \in L^2(G)$, we define

$$(\mathcal{F}_\mathcal{L}[\phi])_\lambda := \sqrt{d_\lambda} \int_G f_\lambda(g)^\dagger \phi(g) \mu_G(dg). \tag{20}$$

The converse map $\mathcal{F}_\mathcal{L}^{-1}$ from $L^2(\hat{G}[\mathcal{L}])$ to $L^2(G)$ is given as

$$\mathcal{F}_\mathcal{L}^{-1}[A](g) := \sum_{\lambda \in \hat{G}[\mathcal{L}]} \sqrt{d_\lambda} \mathrm{Tr}\, f_\lambda(g) A_\lambda \tag{21}$$

for $A = \oplus_\lambda A_\lambda \in L^2(\hat{G}[\mathcal{L}])$ when G is a compact group. When G is not compact group, by choosing a suitable measure $\mu_{\hat{G}[\mathcal{L}]}$ on $\hat{G}[\mathcal{L}]$, the converse map $\mathcal{F}_\mathcal{L}^{-1}$ is characterized as

$$\mathcal{F}_\mathcal{L}^{-1}[A](g) := \int_{\hat{G}[\mathcal{L}]} \sqrt{d_\lambda} \mathrm{Tr}\, f_\lambda(g) A_\lambda \mu_{\hat{G}[\mathcal{L}]}(d\lambda).$$

Since the group G acts on the first space $\mathcal{U}_\lambda$ in the space $\mathcal{U}_\lambda \otimes \mathcal{U}_\lambda^*$, the above representation space $\mathcal{H}$ can be regarded as a subspace of $L^2(\hat{G}[\mathcal{L}])$.

Now, when the factor system of the projective representation f is $\mathcal{L}$, we consider the case when the input state is $|\phi\rangle \in \mathcal{H} \subset L^2(\hat{G}[\mathcal{L}])$ and the measurement is $M_{|\mathcal{I}\rangle\langle\mathcal{I}|}$. Since

$$\langle \mathcal{I}|f(g)|\phi\rangle = \sum_{\lambda \in S} \mathrm{Tr}\, \sqrt{d_\lambda} \mathrm{Tr}\, f_\lambda(g)\phi_\lambda = \mathcal{F}_{\mathcal{L}}^{-1}[\phi](g),$$

the output $\hat{g}$ obeys the probability density function

$$\langle \phi|f(\hat{g})^\dagger|\mathcal{I}\rangle\langle\mathcal{I}|f(\hat{g})|\phi\rangle \mu_G(d\hat{g}) = |\mathcal{F}_{\mathcal{L}}^{-1}[\phi](\hat{g})|^2 \mu_G(d\hat{g}).$$

Therefore, our optimization problem can be described by the terms of converse Fourier transform as follows.

$$\min_{\rho \in \mathcal{S}(\mathcal{H})} \min_{M \in \mathcal{M}_{\mathrm{cov}}(G)} \mathcal{D}_R(\rho, M) = \min_{|\phi\rangle \in \mathcal{H}} \int_G R(e, \hat{g}) |\mathcal{F}_{\mathcal{L}}^{-1}[\phi](\hat{g})|^2 \mu_G(d\hat{g}). \tag{22}$$

4. Finite group

As a typical case, we treat finite groups. It is natural to treat the case

$$R(g, \hat{g}) = \begin{cases} 0 & \text{if } g = \hat{g} \\ 1 & \text{if } g \neq \hat{g} \end{cases}$$

In this case, since the invariant probability measure is $\frac{1}{|G|}$,

$$\mathcal{D}_R(|\phi\rangle\langle\phi|, M_{|\mathcal{I}\rangle\langle\mathcal{I}|}) = 1 - \frac{|\mathcal{F}_{\mathcal{L}}^{-1}[\phi](e)|^2}{|G|}.$$

Hence, it is sufficient to calculate $|\mathcal{F}_{\mathcal{L}}^{-1}[\phi](e)|^2$. Since $\mathcal{V}_\lambda$ is a subspace of $\mathcal{U}_\lambda^*$, $\mathcal{V}_\lambda^*$ can be regarded as subspace of $\mathcal{U}_\lambda$. Now, we denote the projection to the subspace by $P(\mathcal{V}_\lambda^*)$. Hence,

$$\begin{aligned}
&|\mathcal{F}_{\mathcal{L}}^{-1}[\phi](e)|^2 = |\sum_{\lambda \in S} \sqrt{d_\lambda} \mathrm{Tr}\, \phi_\lambda|^2 = |\sum_{\lambda \in S} \sqrt{d_\lambda} \mathrm{Tr}\, \phi_\lambda P(\mathcal{V}_\lambda^*)|^2 \\
\le &(\sum_{\lambda \in S} \sqrt{d_\lambda}^2 \mathrm{Tr}\, P(\mathcal{V}_\lambda^*)^2)(\sum_{\lambda \in S} \mathrm{Tr}\, \phi_\lambda^\dagger \phi_\lambda) = \sum_{\lambda \in S} d_\lambda \dim \mathcal{V}_\lambda.
\end{aligned}$$

Further, the equality holds when $\phi_\lambda = \frac{1}{\sum_{\lambda \in S} d_\lambda \dim \mathcal{V}_\lambda} \sqrt{d_\lambda} P(\mathcal{V}_\lambda^*)$. Thus, we can recover the existing result[7, 10, 15]

$$\min_\phi \mathcal{D}_R(|\phi\rangle\langle\phi|, M_{|\mathcal{I}\rangle\langle\mathcal{I}|}) = 1 - \frac{\sum_{\lambda \in S} d_\lambda \dim \mathcal{V}_\lambda}{|G|}.$$

5. Compact case

When G is a compact group, we consider the case when the error function R is written by using irreducible characters $\{\chi_\lambda\}_{\lambda \in \hat{G}[\mathcal{L}]}$ in the following way.

$$R(g,\hat{g}) = -\sum_{\lambda \in \hat{G}[\mathcal{L}]} a_\lambda (\chi_\lambda(\hat{g}g^{-1}) + \chi_{\lambda^*}(\hat{g}g^{-1})). \quad a_\lambda = a_{\lambda^*} \geq 0 \tag{23}$$

This problem is equivalent with the maximization of the merit function

$$\tilde{R}(g,\hat{g}) = \sum_{\lambda \in \hat{G}[\mathcal{L}]} a_\lambda (\chi_\lambda(\hat{g}g^{-1}) + \chi_{\lambda^*}(\hat{g}g^{-1})). \quad a_\lambda = a_{\lambda^*} \geq 0 \tag{24}$$

For example, in the case of $G = \mathrm{SU}(d)$, as a merit function, we often adopt the Gate fidelity $|\mathrm{Tr}\, g^\dagger \hat{g}|^2$. In the case of $d = 2$, we have $1 + \chi_1(g) = 1 + \chi_{(2,0)}(g) = 1 + \chi_{[2]}(g)$. Then, we obtain the following theorem.

Theorem 1. *For any input pure state $|X\rangle \in L^2(\hat{G}[\mathcal{L}])$, we make a decomposition $|X\rangle = \oplus_{\lambda \in \hat{G}[\mathcal{L}]} c_\lambda |\Phi_\lambda\rangle$ with the conditions $c_\lambda \geq 0$ and $\mathrm{Tr}\, \Phi_\lambda^\dagger \Phi_\lambda = 1$. Choosing an error function R satisfying (23), we obtain*

$$\begin{aligned} \mathcal{D}_R(X) &\geq -\sum_{\lambda,\lambda' \in S} c_\lambda c_{\lambda'} \sum_{\lambda'' \in \hat{G}[\mathcal{L}]} a_{\lambda''} (C_{\lambda,\lambda'^*}^{\lambda''^*} + C_{\lambda,\lambda'^*}^{\lambda''}) \\ &= -\sum_{\lambda'' \in \hat{G}[\mathcal{L}]} a_{\lambda''} \sum_{\lambda,\lambda' \in S} (C_{\lambda'',\lambda}^{\lambda'} + C_{\lambda''^*,\lambda}^{\lambda'}) c_\lambda c_{\lambda'}. \end{aligned} \tag{25}$$

The equality holds when $|\phi\rangle = \oplus_\lambda c_\lambda |\Psi_\lambda\rangle$, where $|\Psi_\lambda\rangle := \frac{1}{\sqrt{d_\lambda}} \sum_{j=1}^{d_\lambda} |\lambda; j; j\rangle$. In this case, the relation $\mathcal{F}_{\mathcal{L}}^{-1}[\phi] = \sum_\lambda c_\lambda \chi_\lambda$ holds.

Therefore, our optimization problem can be reduced to the optimization concerning the choice of $c = (c_\lambda)_{\lambda \in S}$ when our representation space is $\mathcal{H}_S$.

That is, combining Theorem 3, we can recover the following known result[7, 8]. Under the same assumption as 1, we have

$$\begin{aligned} &\min_{\rho \in \mathcal{S}(\mathcal{H}_S), M \in \mathcal{M}(G)} \mathcal{D}_{R,\mu_G}(\rho, M) = \min_{\rho \in \mathcal{S}(\mathcal{H}_S), M \in \mathcal{M}(G)} \mathcal{D}_R(\rho, M) \\ =& \min_{c \in V_S} -\sum_{\lambda,\lambda' \in S} c_\lambda c_{\lambda'} \sum_{\lambda'' \in \hat{G}[\mathcal{L}]} a_{\lambda''} (C_{\lambda,\lambda'^*}^{\lambda''^*} + C_{\lambda,\lambda'^*}^{\lambda''}) \\ =& \min_{c \in V_S} -\sum_{\lambda'' \in \hat{G}[\mathcal{L}]} a_{\lambda''} \sum_{\lambda,\lambda' \in S} (C_{\lambda'',\lambda}^{\lambda'} + C_{\lambda''^*,\lambda}^{\lambda'}) c_\lambda c_{\lambda'}, \end{aligned}$$

where V_S is the set of real vectors $(c_\lambda)_{\lambda \in S}$ satisfying that $c_\lambda \geq 0$ and $\sum_{\lambda \in S} c_\lambda^2 = 1$.

Proof: The relations

$$
\begin{aligned}
&\mathcal{D}_R(\phi) = \int_G R(e,\hat{g}) |\mathcal{F}^{-1}[\phi](\hat{g}^{-1})|^2 \mu_G(d\hat{g}) \\
&= \int_G R(e,\hat{g}) |\sum_{\lambda\in S} \mathrm{Tr}\, f_\lambda(\hat{g}^{-1}) c_\lambda \sqrt{d_\lambda} \Phi_\lambda|^2 \mu_G(d\hat{g}) \\
&= \int_G R(e,\hat{g}) \sum_{\lambda,\lambda'\in S} c_\lambda c_{\lambda'} \mathrm{Tr}\, f_{\lambda^*}(\hat{g}) \otimes f_{\lambda'}(\hat{g}) \sqrt{d_\lambda}\Phi_\lambda \otimes \sqrt{d_{\lambda'}}\Phi_{\lambda'}^\dagger \mu_G(d\hat{g}) \\
&= \sum_{\lambda,\lambda'\in S} c_\lambda c_{\lambda'} \mathrm{Tr}\, [\int_G R(e,\hat{g}) f_{\lambda^*}(\hat{g}) \otimes f_{\lambda'}(\hat{g}) \mu_G(d\hat{g})] \sqrt{d_\lambda}\Phi_\lambda \otimes \sqrt{d_{\lambda'}}\Phi_{\lambda'}^\dagger
\end{aligned} \tag{26}
$$

hold. Using the matrix $\Xi_{\lambda^*,\lambda'} := \int_G -R(e,\hat{g}) f_{\lambda^*}(\hat{g}) \otimes f_{\lambda'}(\hat{g}) \mu_G(d\hat{g})$ and the formula $\int_G \overline{\chi_\lambda(g)} f_{\lambda'}(g) \mu_G(dg) = \frac{\delta_{\lambda,\lambda'}}{d_\lambda} I_\lambda$, we obtain

$$
\begin{aligned}
\Xi_{\lambda^*,\lambda'} &= \int_G \sum_{\lambda''\in\hat{G}[\mathcal{L}]} a_{\lambda''} (\chi_{\lambda''}(\hat{g}) + \chi_{\lambda''^*}(\hat{g})) f_{\lambda^*}(\hat{g}) \otimes f_{\lambda'}(\hat{g}) \mu_G(d\hat{g}) \\
&= \sum_{\lambda''\in\hat{G}[\mathcal{L}]} a_{\lambda''} \int_G (\chi_{\lambda''}(\hat{g}) + \chi_{\lambda''^*}(\hat{g})) f_{\lambda^*}(\hat{g}) \otimes f_{\lambda'}(\hat{g}) \mu_G(d\hat{g}) \\
&= \sum_{\lambda''\in\hat{G}[\mathcal{L}]} \frac{a_{\lambda''}}{d_{\lambda''}} (C_{\lambda^*,\lambda'}^{\lambda''^*} I_{\lambda''^*} + C_{\lambda^*,\lambda'}^{\lambda''} I_{\lambda''}) \ge 0.
\end{aligned}
$$

Hence, applying Schwarz inequality concerning the inner product $\langle A, B\rangle := \mathrm{Tr}\, \Xi_{\lambda^*,\lambda'} A^\dagger B$, to the case $A := I_\lambda \otimes \sqrt{d_{\lambda'}}\Phi_{\lambda'}$, $B := \sqrt{d_\lambda}\Phi_\lambda \otimes I_{\lambda'}$, we obtain

$$
\begin{aligned}
&\mathrm{Tr}\, \Xi_{\lambda^*,\lambda'} \sqrt{d_\lambda}\Phi_\lambda \otimes \sqrt{d_{\lambda'}}\Phi_{\lambda'}^\dagger \\
\le& \sqrt{\mathrm{Tr}\, \Xi_{\lambda^*,\lambda'} d_\lambda \Phi_\lambda^\dagger \Phi_\lambda \otimes I_{\lambda'}} \sqrt{\mathrm{Tr}\, \Xi_{\lambda^*,\lambda'} I_\lambda \otimes d_{\lambda'} \Phi_{\lambda'}^\dagger \Phi_{\lambda'}} \\
=& \sum_{\lambda''\in\hat{G}[\mathcal{L}]} a_{\lambda''} (C_{\lambda,\lambda'^*}^{\lambda''^*} + C_{\lambda,\lambda'^*}^{\lambda''}),
\end{aligned}
$$

where we used the condition $\mathrm{Tr}\, \Phi_\lambda^\dagger \Phi_\lambda = 1$. Combining the above relation with (26), we obtain (25). Due to the equality condition for Schwarz inequality, the equality in (25) holds when $|\phi\rangle = \oplus_\lambda c_\lambda |\Psi_\lambda\rangle$. ∎

Using this formula, we can easily treat the estimation of action of the group $G = \mathrm{U}(1) = \{e^{i\theta} | \theta \in [0, 2\pi]\}$. The one-dimensional unitary representation is characterized by $e^{\lambda i\theta}$ with an integer λ. Hence, $\hat{G}$ is $\mathbb{Z}$, i.e., $L^2(\hat{G}) = L^2(\mathbb{Z})$. That is, the input state is given as a wave function $\varphi(\lambda)$ on the space $L^2(\mathbb{Z})$. The input state φ satisfies $\sum_{\lambda\in\mathbb{Z}} |\varphi(\lambda)|^2 = 1$. Here, we choose the invariant measure $\mu_G(d\hat{g}) = \frac{1}{\sqrt{2\pi}} d\hat{g}$ on G.

The covariant POVM $M_{|\mathcal{I}\rangle\langle\mathcal{I}|}$ is the spectral decomposition of the position operator Q on $L^2(\mathrm{U}(1))$. When the true parameter is 0, the estimate $\hat{g} \in \mathrm{U}(1)$ obeys the distribution $|\mathcal{F}^{-1}[\varphi](-\hat{g})|^2 \frac{1}{\sqrt{2\pi}} d\hat{g}$, which is given by the inverse Fourier transform $\mathcal{F}^{-1}[\varphi]$ of φ.

Since the error can be reduced infinitesimally with the infinite support of the input state, it is natural to restrict the support of φ to $\{k \in \mathbb{Z} | |k| \le n\}$. Now, we treat the error function $1 - \cos(\theta - \hat{\theta}) = 1 - (e^{i(\theta-\hat{\theta})} + e^{i(\theta-\hat{\theta})})/2 = 2\sin^2(\frac{\theta-\hat{\theta}}{2})$. When the input state is $(\varphi(\lambda))_{\lambda=-n}^{n}$, the average error is calculated to

$$1 - \sum_{\lambda=-n}^{n-1} \frac{1}{2}\varphi(\lambda)\varphi(\lambda+1) - \sum_{\lambda=-n+1}^{n} \frac{1}{2}\varphi(\lambda-1)\varphi(\lambda) = 1 - \sum_{\lambda=-n}^{n-1} \varphi(\lambda)\varphi(\lambda+1).$$

Now, we employ the following lemma.

Lemma 1 *[24, 23] The $n \times n$ matrix $P_n := \sum_{j=1}^{n-1} |j\rangle\langle j+1| + |j+1\rangle\langle j|$ has eigenvalues $2\cos\frac{j\pi}{n+1}$ $(j = 1, \ldots, n)$ with the eigenvectors $x^j := (\sin\frac{jk\pi}{n+1})_k$.*

Hence, the above minimum error is $1 - \cos\frac{\pi}{2n+2}$, which is attained by $\varphi(\lambda) = \sin\frac{\pi(\lambda+n+1)}{2n+2}$

As is pointed in [11], the estimation of SU(2) can be treated by using the above calculation for U(1). Then, we can derive the optimal precision of the estimation of SU(2) from the general formula (25) similar to [7].

6. Non-compact case

When G is not compact, the representation space $\mathcal{H} = \oplus_{\lambda\in S}\mathcal{U}_\lambda \otimes \mathcal{V}_\lambda$ might be infinite-dimensional. In this case, it is impossible to prepare an arbitrary initial state. Hence, it is natural to restrict the average energy for the input state. That is, we consider a positive semi-definite self-adjoint operator H_λ on the respective space $\mathcal{U}_\lambda$ and a given constant E, and we assume the condition for the initial state ρ

$$\mathrm{Tr}\,(\bigoplus_{\lambda\in S} H_\lambda \otimes I)\rho \le E^2 \tag{27}$$

When the initial state is given by the pure state $|\phi\rangle = \oplus_\lambda |\phi_\lambda\rangle \in L^2(\hat{G}[\mathcal{L}])$, the above condition can be simplified to

$$\sum_{\lambda\in S} \mathrm{Tr}\, H_\lambda \phi_\lambda \phi_\lambda^\dagger \le E^2. \tag{28}$$

Then, we obtain the following theorem.

Theorem 4 *The relation*

$$\min_{M,\rho}\{\mathcal{D}_R(\rho, M)|(27)\textit{holds.}\} = \min_{\rho}\{\mathcal{D}_R(\rho, M_{|\mathcal{I}\rangle\langle\mathcal{I}|})|(27)\textit{holds.}\} \tag{29}$$

holds. In particular, when

$$E^2 \min_{\phi\in L^2(\hat{G}[\mathcal{L}])}\{\mathcal{D}_R(\rho, M_{|\mathcal{I}\rangle\langle\mathcal{I}|})| \sum_{\lambda\in S} \mathrm{Tr}\, H_\lambda\phi_\lambda\phi_\lambda^\dagger \le E^2, \textit{ and } \|\phi\|_{L^2(\hat{G}[\mathcal{L}])} = 1\} \tag{30}$$

does not depend on E, the relation

$$\begin{aligned} &\min_{\rho}\{\mathcal{D}_R(\rho, M_{|\mathcal{I}\rangle\langle\mathcal{I}|}) | (27) \textit{holds.}\} \\ =& \min_{\phi\in L^2(\hat{G}[\mathcal{L}])} \{\mathcal{D}_R(\phi) | \sum_{\lambda\in S} \operatorname{Tr} H_\lambda \phi_\lambda \phi_\lambda^\dagger \le E^2, \ \textit{and} \ \|\phi\|_{L^2(\hat{G}[\mathcal{L}])} = 1\} \end{aligned} \tag{31}$$

holds.

Therefore, when the above condition holds, it is sufficient to minimize $\mathcal{D}_R(\phi)$ concerning the pure input state $|\phi\rangle$ under the condition (28).
Proof: (29) follows from Theorem 3. We denote (30) by C. Assume that the left hand side of (31) is attained by the state $\sum_i p_i|\phi_i\rangle\langle\phi_i|$. Define $E_i^2 := \langle\phi_i|\bigoplus_{\lambda\in S} H_\lambda|\phi_i\rangle$. Then, $\sum_i p_i E_i^2 \le E^2$. and the left hand side of (31) can be written as $\sum_i p_i \frac{C}{E_i^2}$. Schwarz inequality concerning the vectors $(\sqrt{p_i}E_i)$ and $(\frac{\sqrt{p_i C}}{E_i})$ yields

$$(\sum_i p_i \frac{C}{E_i^2})(\sum_i p_i E_i^2) \ge \sum_i \sqrt{p_i} E_i \frac{\sqrt{p_i C}}{E_i} = \sum_i p_i C = C.$$

Hence, $\sum_i p_i \frac{C}{E_i^2} \ge \frac{C}{E^2}$, which implies (31). ∎

6.1. $\mathbb{R}$. As a typical example of commutative group, we treat the real group $G = \mathbb{R}$. In this case, $\hat{G}$ is also $\mathbb{R}$. That is, since $L^2(\hat{G}) = L^2(\mathbb{R})$, the input state is given as a wave function $\varphi(\lambda)$ on the space $L^2(\mathbb{R})$. Here, we choose the invariant measure $\mu_G(d\hat{g}) = \frac{1}{\sqrt{2\pi}} d\hat{g}$ on G. Then, $\mu_{\hat{G}}(d\lambda) = \frac{1}{\sqrt{2\pi}} d\lambda$. The input state φ satisfies $\int_{\mathbb{R}} |\varphi(\lambda)|^2 \frac{1}{\sqrt{2\pi}} d\lambda = 1$. The covariant POVM $M_{|\mathcal{I}\rangle\langle\mathcal{I}|}$ is the spectral decomposition of the position operator Q on $L^2(\mathbb{R})$. When the true parameter is 0, the estimate $\hat{g} \in \mathbb{R}$ obeys the distribution $|\mathcal{F}^{-1}[\varphi](-\hat{g})|^2 \frac{1}{\sqrt{2\pi}} d\hat{g}$, which is given by the inverse Fourier transform $\mathcal{F}^{-1}[\varphi]$ of φ.

We minimize the average of the square error $(\hat{g} - g)^2$, which is calculated to $\int \hat{g}^2 |\mathcal{F}^{-1}[\varphi](-\hat{g})|^2 d\hat{g}$. In this setting, it is natural to restrict the average energy for input state φ, i.e., we assume that $\int \lambda^2 |\varphi(\lambda)|^2 \frac{1}{\sqrt{2\pi}} d\lambda \le E^2$ for a given constant E. The minimum value is $\frac{1}{4E^2}$, which is attained by the input state $\frac{1}{\sqrt{E}} e^{-\frac{\lambda^2}{4E^2}}$ whose inverse Fourier transform is $\sqrt{2E} e^{-E^2\lambda^2}$.

As another restriction, we assume that the support of the input state $\varphi(\lambda)$ is included in the interval $[-E, E]$. The minimum average of the square error is $\frac{\pi^2}{4E^2}$, which is attained by the input state $\frac{(2\pi)^{1/4}}{\sqrt{E}} \sin \frac{\pi(1+\frac{\lambda}{E})}{2}$ whose inverse Fourier transform is $-\frac{\pi^{3/4}\sqrt{E}}{2^{1/4}} \frac{\cos E\hat{g}}{E^2\hat{g}^2 - \pi^2/4}$. In both settings, the average error behaves in proportion with the inverse of the square of energy.

6.2. $\mathbb{Z}$. Finally, as another typical example of commutative group, we treat the real group $G = \mathbb{Z}$. The one-dimensional unitary representation is characterized by $e^{i\lambda n}$ with a real number $\lambda \in \mathbb{R}$. When the difference between two real numbers

λ and λ' is an integer times of 2π, we obtain $e^{i\lambda n} = e^{i\lambda' n}$. Hence, $\hat{G}$ is $\mathrm{U}(1)$, i.e., $L^2(\hat{G}) = L^2(\mathrm{U}(1))$. That is, the input state is given as a wave function $\varphi(\lambda)$ on the space $L^2(\mathrm{U}(1))$. In this case, the measure on the dual space $\mathrm{U}(1)$ is $\mu_G(d\theta) = \frac{1}{\sqrt{2\pi}} d\theta$.

The input state φ satisfies $\int_{\mathrm{U}(1)} |\varphi(\lambda)|^2 \frac{1}{\sqrt{2\pi}} d\theta = 1$. The covariant POVM $M_{|\mathcal{I}\rangle\langle\mathcal{I}|}$ is the PVM $\{|f_n\rangle\langle f_n|\}$ with $|f_n\rangle := \int_{\mathrm{U}(1)} e^{in\lambda} \frac{1}{(2\pi)^{1/4}} d\lambda$. Here, note that $\{|f_n\rangle\}$ forms a CONS of $L^2(\mathrm{U}(1))$. When the true parameter is 0, the estimate $\hat{g} \in \mathrm{U}(1)$ obeys the distribution $|\mathcal{F}^{-1}[\varphi](-\hat{g})|^2 \frac{1}{\sqrt{2\pi}} d\hat{g}$, which is given by the inverse Fourier transform $\mathcal{F}^{-1}[\varphi]$ of φ.

In this case, when $\varphi(\lambda) = \frac{1}{(2\pi)^{1/4}}$, we have

$$\mathcal{F}^{-1}[\varphi](n) = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0. \end{cases} \tag{32}$$

Hence, if the input $\varphi(\lambda) = \frac{1}{(2\pi)^{1/4}}$ is available, the perfect discrimination is possible.

6.3. Heisenberg representation of $\mathbb{R}^2$. As a typical example of non-commutative representation of a non-compact group, we treat the Heisenberg representation of $\mathbb{R}^2$. Then, we fix the factor system $\mathcal{L}$ defined by the Heisenberg representation. In this case, the representation space is $L^2(\mathbb{R})$. When the multiplicity space is also $L^2(\mathbb{R})$, the input pure state $|\phi\rangle$ can be regarded as an element of $L^2(\mathbb{R})^{\otimes 2}$. Hence, the average of the square error is given as

$$\int_{\mathbb{R}^2} (x_1^2 + x_2^2) |\mathcal{F}_{\mathcal{L}}^{-1}[\phi](-\zeta)|^2 dx_1 dx_2 = \langle \varphi | Q_1^2 + Q_2^2 | \varphi \rangle, \tag{33}$$

where $\zeta = \frac{x_1 + i x_2}{\sqrt{2}}$ and $\varphi := \mathcal{F}_{\mathcal{L}}^{-1}[\phi]$. Now, we consider the energy constraint as follows.

$$\langle \phi | Q^2 + P^2 | \phi \rangle \leq E^2, \tag{34}$$

which can be rewritten as

$$\langle \varphi | (P_2 - \frac{1}{2} Q_1)^2 + (-P_1 - \frac{1}{2} Q_2)^2 | \varphi \rangle \leq E^2. \tag{35}$$

Now, we apply the unitary transformation U corresponding to the the following element of $\mathrm{Sp}(4, \mathbb{R})$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$$

Then, we can convert the above problem to the following: We minimize

$$\langle \varphi | U (Q_1^2 + Q_2^2) U^\dagger | \varphi \rangle \tag{36}$$

under the condition

$$\langle \varphi | U (P_1^2 + P_2^2) U^\dagger | \varphi \rangle \leq E^2. \tag{37}$$

Hence, the minimum value is $\frac{1}{2E^2}$. That is, the average error behaves in proportion with the inverse of the square of energy.

7. Conclusion

We have showed a general formula for the minimum error in the estimation of group action based on the inverse Fourier transform of the input state. Using the obtained formula, we have derived several known formulas, i.e., the maximum discrimination formula in the finite group case and the minimum error formula for the compact group. We have applied the formula to the estimation of a non-compact group action with the energy constraint. Then, we have succeeded in the calculations of the minimum error in the case of $\mathbb{R}$, $\mathbb{Z}$ and the Heisenberg representation. As a result, we have showed that the square speed up phenomena happens in the estimation of shift parameter in the bosonic system. As a future study, we can expect to apply the obtained formula to more general cases and find the square speed up phenomena of the estimation in these cases.

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